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LARGE AMPLITUDE FREE VIBRATION OF CIRCULAR PLATES SUBJECTED TO AERODYNAMIC HEATING

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Abstract—The fundamental non-linear coupled equations of large amplitude free vibration of heated circular plates are derived from the energy equations. The simplified quasilinear, decoupled differential equations for the same case are also obtained by Berger's analysis, that is, by neglecting the second strain invariant of the middle surface from the energy expression.

Both sets of exact and approximate equations are solved separately by a method of successive approximation and also by use of elliptic integrals.

Numerical results are given in graphical form, for both simply supported and clamped circular plates.

NOTATION

d, r_0	thickness and radius of the plate, respectively
t	time
u, v, w	displacement components in the median surface in r, θ and z directions, respectively
D	flexural rigidity
$z(\Theta, t)$	displacement at the centre of the plate
z_1, z_2	absolute values of the maximum and minimum nondimensional amplitudes, respectively
E, G	moduli of elasticity and rigidity, respectively
$\bar{E}$	total energy of the vibrating system
$K(K_1)$	complete elliptical integral of the first kind
T, T^*	linear and non-linear periods, respectively
α, ρ	coefficient of linear thermal expansion and density of the plate material
τ, ξ, μ	non-dimensional time
Θ	temperature change from the initial state
$\bar{\Theta}, \tilde{\Theta}$	mean temperature and temperature moment
ν	Poisson's ratio
χ	Stress function
ω, ω^*	linear and non-linear circular frequencies, respectively
$\bar{\sigma}_{11}, \bar{\sigma}_{22}$	middle surface stresses along r and θ directions respectively
∇^2	$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$
ζ	$= \frac{r}{r_0}$

Subscripts s and c refer to the simply supported plate and the clamped plate, respectively. Superscripts e and a specify the quantity for the cases of the exact method and the approximate method, respectively.

1. INTRODUCTION

PROBLEMS of non-linear vibration of various structural components subjected to aerodynamic heating have become very important from the viewpoint of checking the aeroelastic performance of high speed flying vehicles like missiles and artificial satellites. There are not many works in the literature dealing with the effect of thermal loading on vibration frequencies. Massa [15] has studied the large amplitude, free vibrations of a free circular plate having temperature variation along the radius. The dynamic behaviour of heated rectangular plates with large amplitude was discussed by Sunakawa [2] and Pal [11] for different edge conditions.

Recently the Berger's approximate analysis [3] has been used by many authors, like Nash and Modeer [5], Wah [6] and Gajendar [9] to study nonlinear vibrations of circular and rectangular plates without temperature. In the present work, we study the problem of large amplitude free vibration of heated circular plates, using both the exact analysis and the Berger's approximate analysis. The fundamental equations of motion are deduced both from the energy considerations and by using Berger's analysis [3] which makes the assumption that the effect of second strain invariant in the expression for total energy can be neglected. The equations of motion as obtained from the energy principle are non-linear, coupled differential equations, and those obtained by Berger's analysis are uncoupled, quasi linear differential equations. Both the sets of exact and approximate equations of motion are reduced to a Duffing type equation in the generalized form, whose solution is obtained either by the use of the successive approximation method or by the use of elliptic integrals [2, 8].

Numerical results are presented for both simply supported and clamped plates in graphical form. It is shown that the effects of the temperature change and large amplitude on the period of free vibrations are quite large and can not be ignored, even when the temperature change is small. Further, it is established that Berger's approach yields results sufficient for all practical purposes in engineering.

2. FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS

Let us consider a flat circular plate heated at the upper face. Let us assume that the effects of the internal friction, aerodynamic force and rotary inertia may be neglected.

Let V and T be the potential and kinetic energy of the plate due to bending, stretching and vibration of the plate under the aerodynamic heating; then, due to symmetry of the plate [1], we have,

$$\begin{aligned} V-T = \int_0^{2\pi}\int_0^{r_0}\Bigg[&\frac{D}{2}\left\{\frac{12}{d^2}e_1^2-\frac{24}{d^2}(1-\nu)e_2\right.\\ &\left.+(\nabla^2 w)^2-2(1-\nu)\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}\right\}-\frac{E\alpha}{1-\nu}\{de_1\bar{\Theta}(r)\\ &-(\nabla^2 w)\,d^2\tilde{\Theta}(r)\}+dc_T(\Theta)\\ &-\frac{\rho d}{2}\left\{\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right\}\Bigg]r\,\mathrm{d}r\,\mathrm{d}\theta \end{aligned} \tag{2.1}$$

where

$$\bar{\Theta} = \int_{-d/2}^{d/2}\Theta(r,z)\,\mathrm{d}z, \qquad \tilde{\Theta} = \int_{-d/2}^{d/2} z\Theta(r,z)\,\mathrm{d}z, \tag{2.2}$$

$$\left.\begin{aligned} &e_1 = \bar{E}_{11}+\bar{E}_{22}, \qquad e_2 = \bar{E}_{11}\bar{E}_{22}-\tfrac{1}{4}\bar{E}_{12}^2,\\ &\bar{E}_{11} = \frac{\partial u}{\partial r}+\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2, \qquad \bar{E}_{22} = \frac{u}{r}, \qquad \bar{E}_{12} = 0. \end{aligned}\right\} \tag{2.3}$$

Case 1. Exact equations of motion

Using Hamilton's principle, for extremum of the integral of equation (2.1), we get a set of exact, coupled equations of motion as

$$\frac{1}{12(1-\nu^2)}\left(\frac{d}{r_0}\right)^2\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left[\zeta\frac{\partial}{\partial\zeta}\left\{\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial w'}{\partial\zeta}\right)\right\}\right]$$
$$=\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\chi\frac{\partial w'}{\partial\zeta}\right)-\frac{\alpha}{1-\nu}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial\tilde{\Theta}}{\partial\zeta}\right)-\frac{\rho d^2}{E}\left(\frac{r_0}{d}\right)^2\frac{\partial^2 w'}{\partial t^2} \quad (2.4)$$

$$\frac{\partial}{\partial\zeta}\left\{\frac{1}{\zeta}\frac{\partial}{\partial\zeta}(\zeta\chi)\right\} = -\frac{1}{2\zeta}\left(\frac{d}{r_0}\right)^2\left(\frac{\partial w'}{\partial\zeta}\right)^2-\alpha\frac{\partial\tilde{\Theta}}{\partial\zeta}, \quad (2.5)$$

where

$$\left.\begin{aligned} w' &= \frac{w}{d}, & \zeta &= \frac{r}{r_0} \\ \frac{\bar{\sigma}_{11}}{E} &= r_0\frac{\chi}{r}, & \frac{\bar{\sigma}_{22}}{E} &= r_0\frac{\partial\chi}{\partial r} \end{aligned}\right\} \quad (2.6)$$

and χ is the stress function.

The boundary conditions obtained by using the extremum condition of equation (2.1) are as given in equation (2.7):

For the simply supported plate, we have

$$\left.\begin{aligned} w' &= 0 \\ \frac{\partial^2 w'}{\partial\zeta^2}+\frac{\nu}{\zeta}\frac{\partial w'}{\partial\zeta}+\frac{E\,\mathrm{d}\alpha}{D(1-\nu)}r_0^2\tilde{\Theta} &= 0 \end{aligned}\right\} \text{ at } \zeta = 1. \quad (2.7a)$$

For the clamped plate, we have

$$\left.\begin{aligned} w' &= 0 \\ \frac{\partial w'}{\partial\zeta} &= 0 \end{aligned}\right\} \text{ at } \zeta = 1. \quad (2.7b)$$

Case 2. Berger's approximate equations of motion

By neglecting the second strain invariant e_2 from equation (2.1) and using Hamilton's principle, for extremum of the expression (2.1), we obtain the following set of approximate equations of motion:

$$\frac{1}{12(1-\nu^2)}\left(\frac{d}{r_0}\right)^2\nabla^4 w'-\frac{k^2}{Ed}\nabla^2 w'+\frac{\alpha}{1-\nu}\nabla^2\tilde{\Theta} = -\frac{\rho r_0^2}{E}\frac{\partial^2 w'}{\partial t^2} \quad (2.8)$$

$$\frac{1}{1-\nu^2}e_1-\frac{\alpha\tilde{\Theta}}{1-\nu} = \frac{k^2}{Ed}, \quad (2.9)$$

where k^2 is constant throughout the plate.

The boundary conditions for this case are the same as given in equation (2.7).

The problem is now to obtain the solution of equation of motion (2.4) or (2.8) subject to the boundary conditions (2.7). Assuming the temperature to be symmetrical, the mean and the moment of temperature distributions $\bar{\Theta}$, $\tilde{\Theta}$ are given by

$$\bar{\Theta} = \sum_{i=0,2}^{\infty} \bar{\Theta}_i \zeta^i (i = 0, 2, 4, \ldots \text{even}) \tag{2.10}$$

$$\tilde{\Theta} = \sum_{j=0,2}^{\infty} \tilde{\Theta}_j \zeta^j (j = 0, 2, 4, \ldots \text{even}). \tag{2.11}$$

In the present paper, the vibrations of plates has been analyzed after the change of temperature, and so the temperature change during the vibration has not been considered. For the vibration induced by the thermal shock, the deflection can not be assumed as given in equation (2.13) and therefore, equation (2.4) or (2.8) has to be solved directly. Since it seems to be natural to expect that there exists no remarkable difference between the wave form of the present non-linear vibration and that of the small vibration [16], the lowest mode of vibration is assumed to be the same as the deflection form due to the temperature change only; so, the normal displacement of the plate at any time, $w'(\zeta, \Theta, t)$, is assumed as the sum of the displacement due to the temperature change $F_1(\Theta, \zeta)$ and the amplitude of vibration after the change in temperature $F_2(\zeta, t)_{\Theta=\text{const.}}$, as in the following equation:

$$w'(\zeta, \Theta, t) = F_1(\Theta, \zeta) + F_2(\zeta, t)_{\Theta=\text{const.}} \tag{2.12a}$$

Assuming F_1 and F_2 in the following form

$$F_1(\Theta, \zeta) = z_0(\Theta) w_i(\zeta)$$

and

$$F_2(\zeta, t)_{\Theta=\text{const.}} = \bar{z}(t)_{\Theta=\text{const.}} w_i(\zeta),$$

the expression for the normal displacement w' given by equation (2.12a) becomes

$$w'(\zeta, \Theta, t) = z(\Theta, t) w_i(\zeta) \tag{2.12b}$$

where

$$z(\Theta, t) = z_0(\Theta) + [\bar{z}(t)]_{\Theta=\text{const.}}, \tag{2.13}$$

and

$$w_i(\zeta) = 1 + A_2 \zeta^2 + A_4 \zeta^4, \tag{2.14}$$

in which the constants A_2 and A_4 take the following values, and the subscript i stands for s or c: for simply supported edge:

$$A_2 = -\frac{2(3+\nu)}{5+\nu}, \qquad A_4 = \frac{1+\nu}{5+\nu} \tag{2.15}$$

for clamped edge:

$$A_2 = -2, \qquad A_4 = 1. \tag{2.16}$$

Solving successively the equations of motion (2.4) and (2.5) or (2.8) and (2.9) with the help of equations (2.10)–(2.16) as in [1, 12] we obtain

$$\frac{d^2\bar{z}}{d\tau^2}+(f'_1+3f'_3z_0^2)\bar{z}+3f'_3z_0\bar{z}^2+f'_3\bar{z}^3 = 0 \tag{2.17}$$

and

$$f'_1z_0+f'_3z_0^3 = g \tag{2.18}$$

where

$$f_{1,s}^{1,e} = \frac{1}{a_4}\left\{\left(\frac{r_0}{d}\right)^2 a_2-\frac{1}{12(1-\nu^2)}\right\} \tag{2.19a}$$

$$f_{3,s}^{1,e} = \frac{a_3}{a_4} \tag{2.19b}$$

$$f_{1,c}^{1,e} = 16384\left\{\frac{1}{12}-(1+\nu)\alpha\left(\frac{r_0}{d}\right)^2\left[\frac{5}{72}\left\{\bar{\Theta}_0+(1+\nu)\sum_{i=2}^{\infty}\frac{\bar{\Theta}_i}{i+2}\right\}+8(1-\nu)\sum_{i=2}^{\infty}\frac{(i+5)\bar{\Theta}_i}{(i+2)(i+4)^2(i+6)^2}\right]\right\} \tag{2.19c}$$

$$f_{3,c}^{1,e} = \frac{512}{135}(1+\nu)(173-73\nu) \tag{2.19d}$$

$$f_{1,s}^{1,a} = \frac{16}{L}\left[1-\left\{\frac{1}{12}\left(\frac{2r_0}{d}\right)^2\alpha\sum_{i=0,2}^{\infty}\frac{\bar{\Theta}_i}{i+2}\right\}\times\frac{5\nu^2+40\nu+107}{5+\nu}\right] \tag{2.19e}$$

$$f_{3,s}^{1,a} = \frac{16(\nu^2+10\nu+33)(5\nu^2+40\nu+107)}{9(1+\nu)(5+\nu)^3L} \tag{2.19f}$$

$$f_{1,c}^{1,a} = \frac{36864}{23}\left\{1-\frac{5}{3}\alpha(1+\nu)\left(\frac{r_0}{d}\right)^2\sum_{i=0,2}^{\infty}\frac{\bar{\Theta}_i}{i+2}\right\} \tag{2.19g}$$

$$f_{3,c}^{1,a} = \frac{20480}{23} \tag{2.19h}$$

$$g_s^e = -\left(\frac{r_0}{d}\right)\frac{a_1}{a_4} \tag{2.20a}$$

$$g_c^e = -8192(1+\nu)\alpha\left(\frac{r_0}{d}\right)^2\sum_{j=2}^{\infty}\frac{j}{(j+2)^2}\tilde{\Theta}_j \tag{2.20b}$$

$$g_s^a = \frac{24}{L}\left(\frac{2r_0}{d}\right)^2\alpha\sum_{j=0,2}^{\infty}\frac{(1-\nu)j+4}{(j+2)^2}\tilde{\Theta}_j \tag{2.20c}$$

$$g_c^a = -\frac{221184}{23}\alpha(1+\nu)\left(\frac{r_0}{d}\right)^2\sum_{j=0,2}^{\infty}\frac{j}{(j+2)^2}\tilde{\Theta}_j \tag{2.20d}$$

$$a_1 = \frac{\alpha}{2(1-v^2)}\left\{\tilde{\Theta}_0 + \sum_{j=2}^{\infty} \frac{(1-v)j+4}{(j+2)^2}\tilde{\Theta}_j\right\} \tag{2.21a}$$

$$a_2 = \frac{\alpha}{(1-v^2)(5+v)}\left[\left\{\bar{\Theta}_0 + (1+v)\sum_{i=2}^{\infty}\frac{\bar{\Theta}i}{i+2}\right\}\right.$$

$$\times\left(\frac{5v2+40v+107}{72}\right) + 4(1-v)\sum_{i=2}^{\infty}\frac{\bar{\Theta}_i}{(i+2)(i+4)}$$

$$\left.\times\frac{(i^3+18i^2+108i+214)+2(i+5)(i+8)v+2(i+5)v^2}{(i+4)(i+6)^2}\right] \tag{2.21b}$$

$$a_3 = \frac{1}{4320(1-v^2)(5+v)^3}\{73v^5+895v^4+3198v^3$$

$$-3566v^2-47111v-87249\} \tag{2.21c}$$

$$a_4 = -\frac{(57v^2+574v+1477)}{5760(1+v)(5+v)}\times\frac{r_0^4\rho}{Ed^2} \tag{2.21d}$$

$$L = \frac{23v^2+230v+591}{2304(1+v)(5+v)} \tag{2.22}$$

$$\tau = \frac{1}{(2r_0)^2}\sqrt{\frac{D}{\rho d}}\,t. \tag{2.23}$$

It is to be noted that to evaluate the constant k^2 in equation (2.9), we integrate the equation (2.9) throughout the plate under the condition that there are no in-plane displacements along the edge of the circular plate.

Equations (2.17) and (2.18) represent the equation of dynamic equilibrium and the equation of static equilibrium respectively. The equation of static equilibrium (2.18) was studied in detail by Sunakawa [1] and Pal [12].

The solution of the equation of dynamic equilibrium (2.17) will be given in the present paper.

3. ANALYTICAL SOLUTION

Using the transformation,

$$\xi = \sqrt{(f'_1+3f'_3z_0^2)}\tau \tag{3.1}$$

and thus, changing the independent variable τ in the equation (2.17) to a new variable ξ, we have the following equation:

$$\frac{d^2z}{d\xi^2}+\bar{z}+f_2\bar{z}^2+f_3\bar{z}^3 = 0 \tag{3.2}$$

where

$$\left.\begin{aligned} f_2 &= \frac{3f'_3z_0}{f'_1+3f'_3z_0^2} \\ f_3 &= \frac{f'_3}{f'_1+3f'_3z_0^2} \end{aligned}\right\}. \tag{3.3}$$

For the pre-buckling state, that is when $z_0 = 0$, the term consisting of $\bar{z}^2$ in equation (3.2) vanishes because $f_2 = 0$, and the equation (3.2) becomes simple Duffing type equation which can easily be solved by using elliptic integrals. For the post-buckling state, that is for $z_0 \neq 0$, equation (3.2) becomes the Duffing type equation in the generalized form and can be solved by using elliptic integrals, but these solutions are too complicated and not suitable for technical applications.

Therefore, for the post-buckling state, a method of successive approximation [2, 8] is used for solving the equation (3.2).

Changing the non-dimensional time ξ to μ by the transformation, $\mu = \sqrt{(1+\beta)}\xi$, equation (3.2) becomes

$$(1+\beta)\frac{d^2\bar{z}}{d\mu^2}+\bar{z} = -f_2\bar{z}^2-f_3\bar{z}^3. \tag{3.4}$$

Let z_1 and $-z_2$ be the maximum and minimum value respectively of the displacement $\bar{z}$; then the parameter β and the amplitude $\bar{z}$ are expanded into the power series of z_2 as

$$\beta = -\beta_1 z_2+\beta_2 z_2^2-\beta_3 z_2^3+\beta_4 z_2^4- \dots \tag{3.5}$$

$$\bar{z} = -\eta_1(\mu)z_2+\eta_2(\mu)z_2^2-\eta_3(\mu)z_2^3+ \dots, \tag{3.6}$$

where β_i and η_i, $i = 1, 2, 3, \dots$ are to be determined.

Substituting equations (3.5) and (3.6) in equation (3.4) and equating the coefficients of z_2 and its higher powers, each to zero, we obtain a set of differential equations in $\eta_1, \eta_2, \eta_3, \dots$, with coefficients consisting of $\beta_1, \beta_2, \beta_3, \dots$. Solving these equations by the successive approximation method [2, 8], with the initial conditions

$$\eta_1(0) = 1, \qquad \eta_2(0) = \eta_3(0) = \dots = 0,$$

and

$$\dot{\eta}_1(0) = \dot{\eta}_2(0) = \dot{\eta}_3(0) = \dots = 0,$$

we can get the values of $\beta_1, \beta_2, \beta_3, \dots$ and $\eta_1, \eta_2, \eta_3, \dots$ after eliminating the possibility of resonance. Thus, finally the solution of equation (3.4) is obtained as

$$\begin{aligned}\bar{z} = {} & [-\tfrac{1}{2}f_2z_2^2+\tfrac{1}{3}f_2^2z_2^3-(\tfrac{25}{48}f_2^3-\tfrac{21}{32}f_2f_3)z_2^4+(\tfrac{25}{36}f_2^4-\tfrac{29}{24}f_2^2f_3)z_2^5-\dots] \\ & +[-z_2+\tfrac{1}{3}f_2z_2^2-(\tfrac{29}{144}f_2^2-\tfrac{1}{32}f_3)z_2^3+(\tfrac{119}{432}f_2^3-\tfrac{35}{96}f_2f_3)z_2^4-(\tfrac{6971}{20736}f_2^4-\tfrac{1475}{2304}f_2^2f_3 \\ & +\tfrac{23}{1024}f_3^2)z_2^5+\dots]\cos\mu+[\tfrac{1}{6}f_2z_2^2-\tfrac{1}{9}f_2^2z_2^3+(\tfrac{2}{9}f_2^3-\tfrac{1}{3}f_2f_3)z_2^4-(\tfrac{8}{27}f_2^4-\tfrac{5}{9}f_2^2f_3)z_2^5+\dots]\cos 2\mu \\ & +[-(\tfrac{1}{48}f_2^2+\tfrac{1}{32}f_3)z_2^3+(\tfrac{1}{48}f_2^3+\tfrac{1}{32}f_2f_3)z_2^4-(\tfrac{31}{576}f_2^4-\tfrac{11}{384}f_2^2f_3-\tfrac{3}{128}f_3^2)z_2^5+\dots]\cos 3\mu \\ & +[(\tfrac{1}{432}f_2^3+\tfrac{1}{96}f_2f_3)z_2^4-(\tfrac{1}{324}f_2^4+\tfrac{1}{72}f_2^2f_3)z_2^5+\dots]\cos 4\mu \\ & +[-(\tfrac{5}{20736}f_2^4+\tfrac{5}{2304}f_2^2f_3+\tfrac{1}{1024}f_3^2)z_2^5+\dots]\cos 5\mu+\dots \\ & +\dots.\end{aligned} \tag{3.7}$$

The circular frequency $\omega^*(t)$ and the period of motion $T^*(t)$ are given as

$$\omega^*(t) = \frac{\sqrt{(f'_1+3f'_3z_0^2)}}{(2r_0)^2}\sqrt{\left(\frac{D}{\rho d}\right)}\sqrt{(1+\beta)}$$

and

$$T^*(t) = \frac{2\pi(2r_0)^2}{\sqrt{(f'_1+3f'_3z_0^2)}}\sqrt{\left(\frac{\rho d}{D}\right)}[1+(\tfrac{5}{12}f_2^2-\tfrac{3}{8}f_3)z_2^2 - (\tfrac{5}{18}f_2^3-\tfrac{1}{4}f_2f_3)z_2^3+(\tfrac{385}{576}f_2^4-\tfrac{275}{192}f_2^2f_3+\tfrac{57}{256}f_3^2)z_2^4-\ldots]. \tag{3.8}$$

Equation (3.7) and (3.8) give, respectively, the circular frequency and the period of the large amplitude free vibration of circular plates subjected to aero-dynamic heating. To get the relation between the maximum and minimum values of amplitude z_1 and $-z_2$ we integrate the equation (3.2) once and use the condition that

$$\frac{d\bar{z}}{d\xi} = 0 \quad \text{at } \bar{z} = z_1, \qquad -z_2,$$

which gives,

$$z_1^2(1+\tfrac{2}{3}f_2z_1+\tfrac{1}{2}f_3z_1^2) = z_2^2(1-\tfrac{2}{3}f_2z_2+\tfrac{1}{2}f_3z_2^2) = 2\bar{E}, \tag{3.9}$$

where $\bar{E}$ is the total energy of the vibrating system. Equation (3.9) gives the relation between the maximum and minimum values of amplitude z_1 and $-z_2$.

For the pre-buckling state, equation (3.2) is reduced to

$$\frac{d^2\bar{z}}{d\tau^2}+f'_1\bar{z}+f'_3\bar{z}^3 = 0. \tag{3.10}$$

Through the energy integral [2], equation (3.10) gives

$$\tau = \pm\frac{1}{\sqrt{(f'_1+f'_3z_2^2)}}\int_0^{\phi}\frac{d\phi}{\sqrt{(1-K_1^2\cos^2\phi)}},$$

where

$$K_1^2 = \frac{f'_3z_2^2}{2(f'_1+f'_3z_2^2)}, \qquad \frac{\bar{z}}{z_2} = \sin\phi.$$

The sign of the above integral is taken to be positive or negative according as $\bar{z}$ increases or decreases with the increase of τ.

Then the period of vibration is

$$T^*(t) = \frac{4(2r_0)^2}{\sqrt{(f'_1+f'_3z_2^2)}}\sqrt{\left(\frac{\rho d}{D}\right)}K(K_1) \tag{3.11}$$

where $K(K_1)$ is the complete elliptic integral of the first kind.

4. NUMERICAL EXAMPLES

Let us assume that the circular plates, simply supported or clamped along the edge, are subjected to the following change in temperature

$$\left.\begin{aligned}\Theta = \bar{\Theta} &= [\Theta_0+\Theta_1(1-\zeta^2)]\\ \tilde{\Theta} &= 0 \quad \text{that is } g = 0\end{aligned}\right\}. \tag{4.1}$$

Taking $\nu = \frac{1}{3}$, equations (2.19)–(2.22) with the help of equations (2.10), (2.11), (2.18), (3.3) and (4.1) are reduced to

$$f_{1,s}^{1,e} = \frac{6144}{157}\left[1-\frac{1}{18}\left(\frac{2r_0}{d}\right)^2\alpha\Theta_1\left\{\frac{931}{96}+17\frac{\Theta_0}{\Theta_1}\right\}\right] \tag{4.2a}$$

$$f_{3,s}^{1,e} = \frac{27}{37680}\left(87249+\frac{47111}{3}+\frac{3566}{9}-\frac{3198}{27}-\frac{895}{81}-\frac{73}{243}\right) \tag{4.2b}$$

$$f_{1,c}^{1,e} = 4096\left\{\frac{1}{3}-\frac{1}{1296}\left(\frac{2r_0}{d}\right)^2\alpha\Theta_1\left(120\frac{\Theta_0}{\Theta_1}+73\right)\right\} \tag{4.2c}$$

$$f_{3,c}^{1,e} = \frac{512\times 1784}{1215} \tag{4.2d}$$

$$f_{1,s}^{1,a} = \frac{147456}{377}\left\{1-\frac{17}{36}\left(\frac{2r_0}{d}\right)^2\alpha\Theta_1\left(1+2\frac{\Theta_0}{\Theta_1}\right)\right\} \tag{4.2e}$$

$$f_{3,s}^{1,a} = \frac{697\times 512}{377} \tag{4.2f}$$

$$f_{1,c}^{1,a} = \frac{36864}{23}\left\{1-\frac{5}{36}\left(\frac{2r_0}{d}\right)^2\alpha\Theta_1\left(1+2\frac{\Theta_0}{\Theta_1}\right)\right\} \tag{4.2g}$$

$$f_{1,c}^{1,a} = \frac{20480}{23} \tag{4.2h}$$

$$g_s^e = \frac{3032}{1413}\left(\frac{2r_0}{d}\right)^2\alpha\Theta_1\left(1+1{\cdot}5\frac{\Theta_0}{\Theta_1}\right) \tag{4.3a}$$

$$g_c^e = \frac{512}{27}\left(\frac{2r_0}{d}\right)^2\alpha\Theta_1 \tag{4.3b}$$

$$g_s^a = \frac{8192}{377}\left(\frac{2r_0}{d}\right)^2\alpha\Theta_1\left(1+1{\cdot}5\frac{\Theta_0}{\Theta_1}\right) \tag{4.3c}$$

$$g_c^a = \frac{512}{23}\left(\frac{2r_0}{d}\right)^2\alpha\Theta_1. \tag{4.3d}$$

The critical temperature (Θ_1)cr. and the deflection at the centre of the plate after buckling are obtained, with Θ_0/Θ_1 as a parameter, for both exact and approximate cases from equation (2.18) and compared as shown in Fig. 2 for different edge conditions, namely, simply supported and clamped plates.

The behaviour of the plate for pre-buckling and post-buckling state is discussed below:

Case 1

$$\Theta_1 < (\Theta_1)\text{cr.} \qquad (z_0 = 0)$$

The variation of the ratio of the non-linear and linear periods, T^*/T with the amplitude for different temperatures is studied from equation (3.8) and the comparison between the exact and the approximate analyses is shown in Fig. 3, for different boundary conditions. The change of circular frequency with the amplitude is also compared and shown in Figs. 4 and 5 using the temperature as a parameter. It is seen that the circular frequency decreases with the increase of temperature for the pre-buckling state.

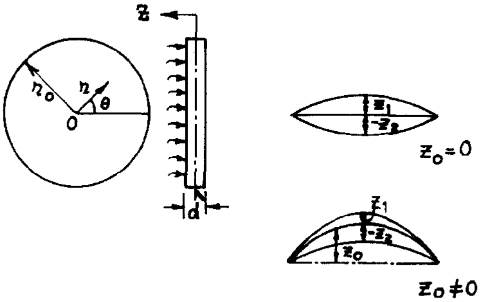

FIG. 1. Circular plate.

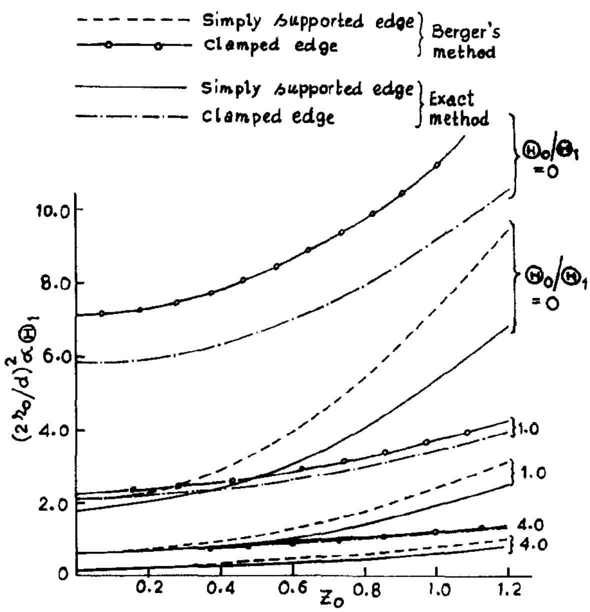

FIG. 2. Relation between temperature rise and deflection at the centre of the plate.

Case 2

$$\Theta_1 > (\Theta_1)\text{cr}. \qquad (z_0 \neq 0)$$

For the post-buckling state, the change of circular frequency with the amplitude, taking the temperature as a parameter, is obtained with the help of equations (3.8), (2.18) and (4.1), for both the exact and approximate analyses and shown in Figs. 4 and 5 for different edge conditions.

The results show that the circular frequency increases with the increase of temperature for the post-buckling state and decreases with the increase of temperature for the pre-buckling state.

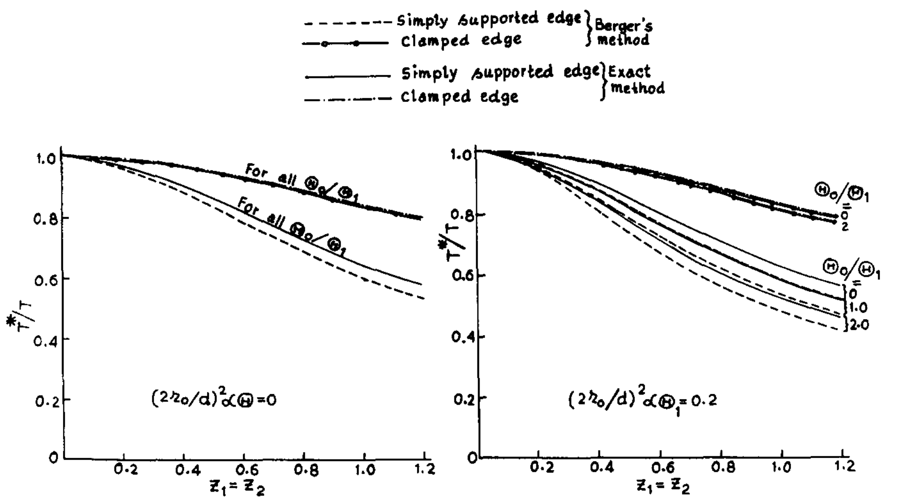

FIG. 3. Influence of large amplitude on period of vibration.

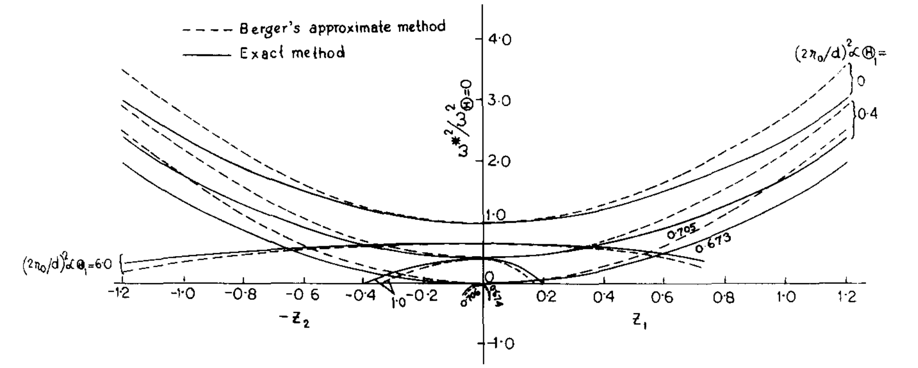

FIG. 4. Variation of frequencies of vibration of plate with temperature rise and large amplitude, simply supported edge, $\Theta_0/\Theta_1 = 1{\cdot}0$.

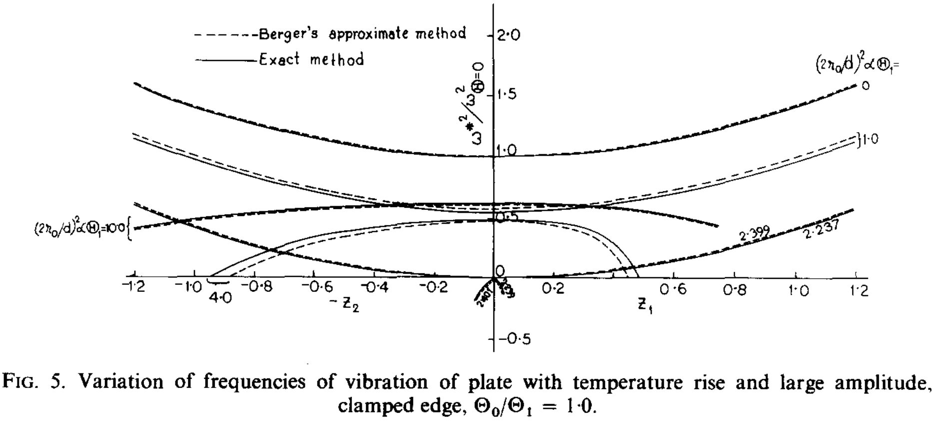

FIG. 5. Variation of frequencies of vibration of plate with temperature rise and large amplitude, clamped edge, $\Theta_0/\Theta_1 = 1{\cdot}0$.

Figures 4 and 5 show that the circular frequency increases with the increase of amplitude for the pre-buckling state and reaches a minimum at the buckling temperature, but decreases with the increase of amplitude for the post-buckling state until a snap through phenomenon takes place, and it is seen that the effect of temperature on the vibration can not be ignored.

During the post buckling state, the plate attains a certain maximum absolute value of amplitude, for a certain temperature. At this maximum absolute value of amplitude, the plate suddenly starts deflecting in the opposite side, that is, the snap through phenomenon takes place. This maximum absolute value of the amplitude corresponds to the point where $\Theta_1 > (\Theta_1)$cr., that is where a certain particular temperature after buckling, intersects the abscissa. After the occurrence of such a phenomenon, the plate may start to vibrate about the new position of equilibrium.

It is seen that the results obtained from Berger's approximate analysis agree closely with those obtained from the exact analysis, which indicates the validity of Berger's analysis for all practical purposes.

5. CONCLUSION

The non-linear natural vibration of heated circular plates, simply supported or clamped, has been studied by the use of an exact method as well as an approximate method. For the exact method, the fundamental equations of motion are derived from the energy equation, and since these equations are non-linear and coupled, the solutions are difficult to obtain. But in the case of the approximate method, simplified quasilinear, decoupled equations of motion for non-linear vibrations of plates subjected to heating are obtained by Berger's analysis, and they can be easily solved by a successive approximation method, or by the use of elliptic integrals as indicated in the present paper.

The results obtained by Berger's approximate analysis show that they are in good agreement with those obtained by an exact analysis, which confirms the validity of Berger's analysis for the thermal stress.

An analysis, such as presented here, will be of considerable importance in supersonic airplanes, missiles and satellites, whenever the vibration has no small effect on the aero- and thermo-elastic problems. It appears also a reasonable conclusion that the Berger's approach presented in this paper is likely to yield results entirely adequate for many technical purposes.

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Абстракт—Определяются, из уравнений энергии, основные, нелинейные, сопряженные уравения свободных колебаний большой амплитуды для нагретых, круглых пларргинок. Упрощенные, квази-линейные, распряженные дифференциальные уравнения для того-же случая получаются также методом анализа Бергера, то есть, пренебрегая вторым инвариантом деформации серединной поверхности в выражении энергии.

Решаются, отдельно, обе системы точных и приближенных уравнений путем применения метода последовательных приближений, а также используя эллиптические интегралы.

Даются численные результаты в форме графиков, так для свободно опертых как и защемленных круглых пластинок.